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Statistics for small numbers of events with background

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The confidence level CL for an upper limit μ^{sup} resulting from the observation of a given number of events N is intuitively defined by the following relation:

$$CL = \int_0^{\mu^{sup}} P(N;\mu)d\mu \qquad (1)$$

the Poisson function:

$$P(N;\mu) = e^{-\mu}\frac{\mu^N}{N!} \qquad (2)$$

expressing the degree of likeliness that the parameter characterizing the physical process being investigated has the particular value μ, considering the result of the measurement which is N. From experiments repeated in the same conditions, the μ parameter happens to be the average number of observed events.

From a known property of the Poisson function, formula (1) can be rewritten as follows:

$$CL = 1 - \sum_{n=0}^{N} P(n;\mu_s^{sup}) \qquad (3)$$

The Poisson function $P(N;\mu)$, considered as likelihood function or likelihood amplitude is normalized to 1 as it should for relation (1) to have its full meaning:

$$\int_0^{\infty} P(N;\mu)d\mu = 1 \qquad (4)$$

i.e. integrating over all possible values of μ, from 0 to ∞.

In the case where there is signal in the presence of background, let us consider that the average number of background events, μ_b , is known and that the average number of signal events, μ_s, is the parameter one wants to evaluate. If the background and signal events are undistinguishable, the μ parameter characterizing the Poisson distribution of events is equal to the sum ($\mu_s + \mu_b$). Indeed, according to a special property of the Poisson function, the probability of observing N events is equal to the sum of the probabilities of finding n_i signal events and ($N-n_i$) background events, for all the values of n_i from 0 to N :

$$P(N;\mu_s + \mu_b) = \sum_{n_i=0}^{n_i=N} P(n_i;\mu_s)P(N-n_i;\mu_b) \qquad (5)$$

The following Table gives as an example the probabilities for the various possible configurations for $\mu_s = 1$, $\mu_b = 3$ and N = 3

Table 1:

n_s	n_b	$P(n_s;\mu_s)\,P(n_b;\mu_b)$
0	3	8.24%
1	2	8.24%
2	1	2.75%
3	0	0.31%
Total =		19.54%
P(3; 1 + 3) =		19.54%

$P(N;\mu_s+\mu_b)$ may therefore be considered as expressing the likeliness that the true value of the signal parameter takes a particular value μ_s . However as such, it is not normalized to1 with respect to μ_s :

$$\int_0^\infty P(N;\mu_s+\mu_b)d\mu_s = \int_{\mu_b}^\infty P(N;\mu')d\mu' \qquad (6)$$

Therefore, the normalized likelihood amplitude for μ_s should be:

$$P_{norm}(N,\mu_b;\mu_s) = \frac{P(N;\mu_s+\mu_b)}{\int_{\mu_b}^\infty P(N;\mu')d\mu'} \qquad (7)$$

and the signal upper limit μ_s^{sup} at the confidence level CL would be obtained according to the defining relation (1) as follows:

$$CL = \int_0^{\mu_s^{sup}} P_{norm}(N,\mu_b;\mu_s)d\mu_s \qquad (8)$$

that is:

$$CL = \frac{\int_0^{\mu_s^{sup}} P(N;\mu_s+\mu_b)d\mu_s}{\int_{\mu_b}^\infty P(N;\mu')d\mu'} \qquad (8')$$

or, given that the likelihood amplitude (7) is normalized to 1:

$$CL = 1 - \frac{\int_{\mu_s^{sup}}^\infty P(N;\mu_s+\mu_b)d\mu_s}{\int_{\mu_b}^\infty P(N;\mu')d\mu'} = 1 - \frac{\int_{\mu_b+\mu_s^{sup}}^\infty P(N;\mu')d\mu'}{\int_{\mu_b}^\infty P(N;\mu')d\mu'} \qquad (9)$$

Using the following property of the Poisson function:

$$\int_\mu^\infty P(N;\mu')d\mu' = \sum_{n=0}^{N} P(n;\mu) \qquad (10)$$

relation (9) becomes:

$$CL = 1 - \frac{\sum_{n=0}^{N} P(n;\mu_s^{sup}+\mu_b)}{\sum_{n=0}^{N} P(n;\mu_b)} \qquad (11)$$

which is the currently used formula as recommended by the CERN "Particle Data Group" (see for example Booklet July 1996, p. 211).

So, in practice, the signal upper limit μ_s^{sup} at a given confidence level CL can be evaluated equally from relation (8) or from relation (11)

It to be remarked that for N = 0, the signal upper limit μ_s^{sup} is independent of the background parameter μ_b. Indeed, with N = 0 in (11), one gets:

$$CL = 1 - \frac{P(0;\mu_s^{\sup} + \mu_b)}{P(0;\mu_b)} = 1 - \frac{e^{-\left(\mu_s^{\sup} + \mu_b\right)}}{e^{-\mu_b}} = 1 - e^{-\mu_s^{\sup}} \qquad (12)$$

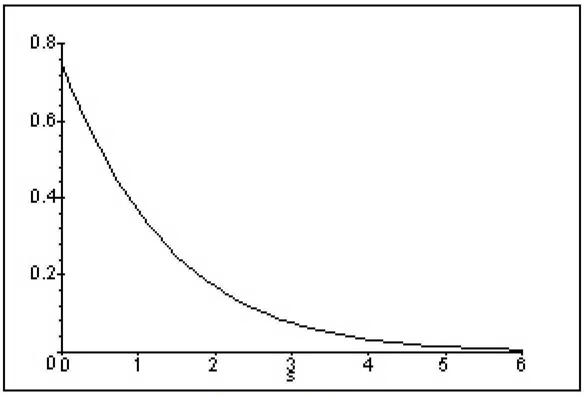
Upper limits and confidence intervals.

- As an example, the Table2 gives the signal upper limits $\mu_s^{\sup}$ evaluated at 90% and 95% confidence level for a background $\mu_b = 3$ and for values of $N \le 8$:

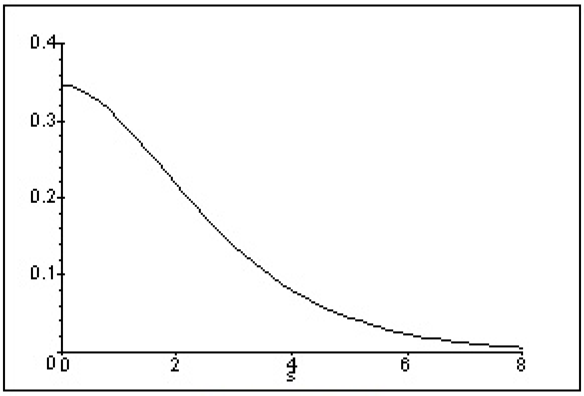
Table2:

N / CL	$\mu_s^{\sup}$	
	90%	95%
0	2.30	3.00
1	2.84	3.64
2	3.52	4.44
3	4.36	5.40
4	5.34	6.48
5	6.43	7.66
6	7.60	8.90
7	8.80	10.17
8	10.00	11.42

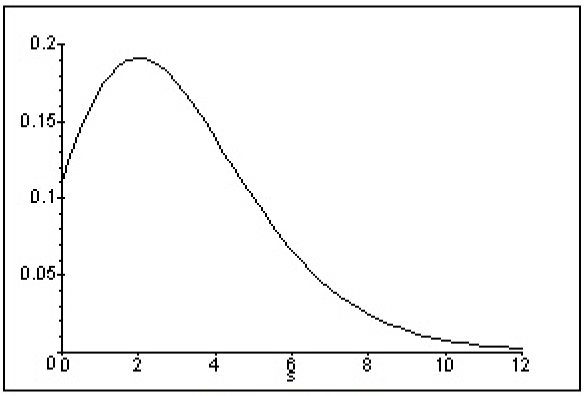
For values of N up to 8, considering confidence intervals from 0 to an upper limit $\mu_s^{\sup}$ can be justified by the shape of the normalized likelihood amplitude $P_{norm}(N,\mu_b;\mu_s)$ (8) as function of μ_s.



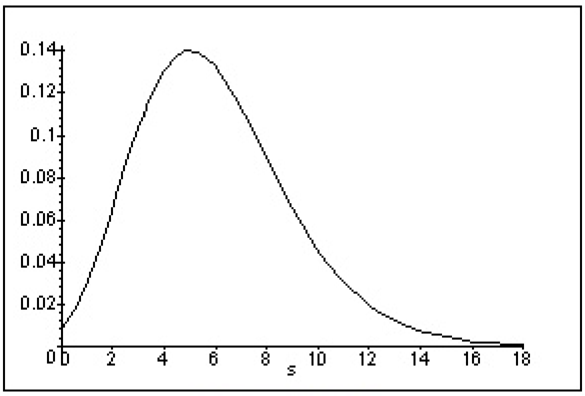
$P_{norm}(1,3;s)$



$P_{norm}(3,3;s)$



$P_{norm}(5,3;s)$

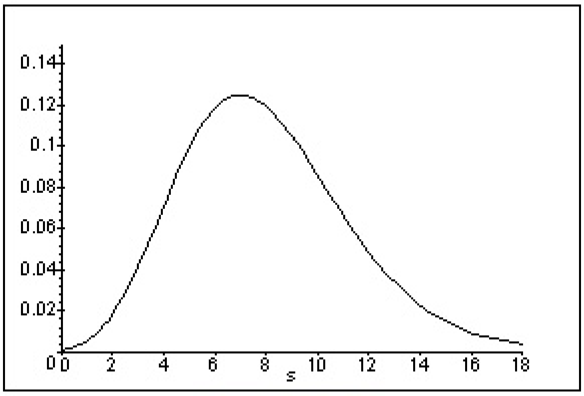


$P_{norm}(8,3;s)$

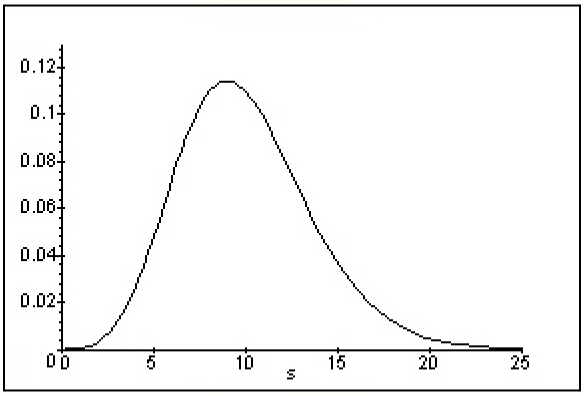
- For values of N greater than 8, the amplitude takes a bell-like shape shifting toward larger and larger values of μ_s. In these conditions, a confidence interval starting at 0 contains a region which go increasing where it is not very likely that the true value of μ_s may be expected. It makes therefore more sense to define a confidence interval between two limits, $\mu_s^{\inf}$ et $\mu_s^{\sup}$, such that the integral of $P_{norm}(N, \mu_b; \mu_s)$ is equal to the confidence level CL:

$$\int_{\mu_s^{\inf}}^{\mu_s^{\sup}} P_{norm}(N, \mu_b; \mu_s) d\mu_s = CL \qquad (13)$$

The choice of the limit $\mu_s^{\inf}$ is arbitrary.



$P_{norm}(10,3;s)$



$P_{norm}(12,3;s)$

In the following, it is chosen so as to define the confidence interval in the most accurate way, i.e. by requiring the difference ($\mu_s^{\inf}$ - $\mu_s^{\sup}$) to be a minimum.

Table 3 gives also as an example the signal limits evaluated at 90% and 95% confidence level for a background μ_b = 3 and for values of N from 6 to 12.

Table 3:

N / (lim.)$_{CL}$	CL = 90%		CL = 95%	
	$\mu_s^{\inf}$	$\mu_s^{\sup}$	$\mu_s^{\inf}$	$\mu_s^{\sup}$
6	0 (0)	7.60 (7.60)	0 (0)	8.90 (8.90)
7	0.55 (0)	9.18 (8.8)	0.10 (0)	10.26 (10.17)
8	1.2 (0)	10.60 (10.00)	0.70 (0)	11.88 (11.42)
9	1.9	11.92	1.30	13.26
10	2.6	13.16	2.00	14.62
11	3.35	14.42	2.70	15.92
12	4.15	15.70	3.40	17.72

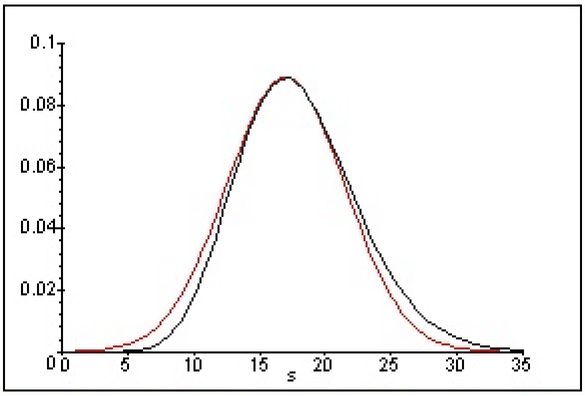
The numbers between parentheses define the confidence intervals starting from the value μ_s = 0 ($\mu_s^{\sup}$ as in Table 2). One can see that for N up to 6, the confidence interval from 0 to $\mu_s^{\sup}$ is the smallest.

The so determined confidence intervals are somewhat different than those which have been evaluated by Feldman and Cousin either from the standard method or from their ordering technique (PRD 57, 3873, 1998). They happen to be slightly shifted downwards. However, the upper limits of $\mu_s^{\sup}$ with the confidence interval starting at 0 are rather different than those which have been determined in an indirect way by these authors: they are higher for $N \leq 3$, smaller above that value.

- For much larger values of N, the normalized likelihood amplitude $P_{norm}(N, \mu_b; \mu_s)$ (7),

is approaching a gaussian function characterized by a mean value equal to ($N-\mu_b$) and by a standard deviation $\rho = \sqrt{N}$ and one is then led to use gaussian statistics.

As an example, the figure below shows the normalized likelihood amplitude $P_{norm}(N, \mu_b; \mu_s)$ for N = 20 and again $\mu_b = 3$ for comparison with the corresponding gaussian function.



Black : $P_{norm}(20,3;s)$ - Red: Gauss ($\sigma=\sqrt{N}$)